Dr. Thorsten Hertl

Graduate Seminar, Semester 2 2025
Rational Homotopy Theory

On of the main goals in algebraic topology is to determine the homotopy type of a topological space X and to determine the set $[X, Y]$ of homotopy classes of maps. In general, it is practically impossible to give a complete description of $[X, Y]$, even in concrete cases of interest.
Informally, one reason for this difficulty, besided the lack of algebraic structures on $[X, Y]$, is the existence of torsion, which completely disappears if one restricts their attention to rational space, which are simply connected spaces whose (higher) homotopy groups are rational vector spaces. For example, while the homotopy groups of spheres have not been determined completely yet, the rational homotopy groups of spheres are known since the '50s of the last century.
Even better, we will see in this seminar that the rational homotopy groups of a simply connected space is completely determined by an algebraic object, a certain commutative, graded, differential algebra (cgda) to be precise, and that the homotopy classes of two rational spaces are given by equivalence classes of homomorphism between the corresponding algebraic objects. It turns out that these algebraic objects can be computed with reasonable effort, which allows for powerful applications in geometry, topology, and perhaps even data science.

The seminar is partitioned in two parts. The first one focusses on the theoretical foundations of rational homotopy theory and is given by the lecturer. There we will see, after a short reminder of the topological preliminaries, what a rational space is, how to localise a given (simply connected) topological space along the rational numbers, and how a topological space generates these algebraic objects. We then will turn our focus to the algebraic side, study the theory of cgdas, and see that every topological space has a minimal model that is unique up to isomorphism and only depends on the homotopy type of the topological space.
The second part focusses more on geometric and topological application of these algebraic models. This part is less consecutive than the first part and more example oriented. The concrete topics will depend on the interest of the audience, which are encouraged to give a talk themselves.

The seminar is intended for Master and PhD students with interest in topology and geometry with an interest in the concrete calculation of topological invariants and their application in geometry and topology. However, every student with mild previous knowledge in topology will be able to attend this seminar. The main source of this seminar is the book [1], which emphasises the discussion of examples over detailed proofs. A more detailed account for the underlying theory can be found in [2], a nice survey are [3] and [4].

(Preliminary) List of Talks

Part I - Foundations

Talk 01: Basics on CW complexes, - Thorsten Hertl
Definition of CW complexes, homotopy groups, Whiteheads theorem, (Serre) fibrations, localisation and rationalisation of CW complexes, Postnikov decomposition, CW approximation.

Talk 02: Basics simplicial sets, - Thorsten Hertl
Definition of simplicial sets and categories; the singular set of a topological space, adjunction $\mathsf{Top} \leftrightarrows \mathsf{sSet}$, Kan complexes, A_{PL} of a simplicial set, Crash course on model categories

Talk 03 - 05: Rational Homotopy Theory I, II, III, - Thorsten Hertl, Chapter 2.1 - 2.3 in [1]
Definition of cdga, (relative) Sullivan models, minimal models; uniqueness of minimal models; notion of homotopy between cdga, Whiteheads theorem [Theorem 2.17], lifting and extension of homotopies [Proposition 2.11 and 2.22], existence and uniqueness of (relative) minimal models

Talk 06: Realisation of Topoloical Spaces, - Thorsten Hertl, Chapter 2.4 + 2.6 in [1]
$A_{PL}(X)$, adjunction $\mathsf{CGDA}_Q^{op} \leftrightarrows \mathsf{sSet}_{\leq 1}$ and that it induces an equivalence between homotopy categories, rational homotopy groups of topological spaces in terms of algebra, Whitehead-product and algebraic counter part, examples of minimal models for various spaces.

Part II - Applications

Talk 07: Loop spaces and application to geodesics, - N.N, Chapter 5 in [1]
Present how one can derive a minimal model for the loop space ΩX from a minimal model of X. As an application, show that most Riemannian manifolds have a lot of geodesics.

Talk 08: Rational Models of Classifying Spaces, - N.N, Chapter 3.3 in [1]
T.B.D

Talk 09: Rational Models for Biquotients, - N.N, Chapter 3.4 in [1]
T.B.D

Talk 10: Formality and Kähler manifolds, - N.N, Chapter 2.7 and 4 in [1]
Here is a lot of material to cover, and one can be quite sketchy with the proofs. Depending how the previous knowledge, you may need to invest some time on the definition of the complex de Rham complex and its natural bigrading.

Talk 11: Configuration Spaces, - N.N, Chapter 9.1 in [1]
T.B.D.

Talk 12: Rational Models of $B\mathrm{hAut}$, - N.N, ?? The original proof due to Sulliavan is sketchy at best. Here, we also need to work with Lie-models, which only have been discussed shortly. Maybe present only if there is enough interest.

Literatur

[1] Yves Félix, John Oprea, and Daniel Tanré, *Algebraic models in geometry*, Oxford Graduate Texts in Mathematics, vol. 17, Oxford University Press, Oxford, 2008. MR2403898